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Absolute continuity of the integrated density of states of the quantum Lorentz gas for a class of repulsive potentials

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Abstract. For a class of repulsive potentials, for instance of the type $\varphi(x)=|x|^{-\kappa}[1+x^2]^{-\lambda}$ (κ, λ positive, in a certain range) in the random Schrödinger operators $H(\omega)=p^2+V(x,\omega)=p^2+\Sigma_j\,\varphi(x-x_j)$, acting in $L^2(\mathbb{R}^d)$, with Poisson distributed x_js (the quantum Lorentz gas), we show that the integrated density of states $N(E)$ is absolutely continuous for $E>\zeta\rho\langle\varphi\rangle$. Here is $\langle\varphi\rangle$ the integral of φ over $\mathbb{R}^d$, ρ the averaged density of points x_j and $\zeta>0$ depends on φ and d. In the above example, $\zeta=(d/\kappa)^2$. Our method makes use of a Fock space representation for the Poisson random system, recently developed by Maassen and the author. Within this Fock space formalism the Mourre commutator method is then employed to obtain the announced result.

1. Introduction

Although much is known about asymptotic properties, such as Lifshitz tails, of the integrated density of states $N(E)$ for random Schrödinger systems (Kirsch 1989, Carmona and Lacroix 1990) the situation seems to be quite different with respect to its absolute continuity (differentiability) beyond the one-dimensional situation (Carmona and Lacroix 1990, preface). In the present work we present some partial results concerning this property for the quantum Lorentz gas, i.e. a single Schrödinger particle moving in a background of randomly placed identical scattering centres, distributed according to a Poisson law. Thus we are dealing with a family of random Hamiltonians

$$H(\mu)=p^2+\sum_j \varphi(x-x_j)=p^2+\int \mu(\mathrm{d}y)\varphi(x-y)=p^2+\langle\varphi_x^-,\mu\rangle \qquad (1.1)$$

acting in $\mathscr{H}=L^2(\mathbb{R}^d)$, where the x_js are Poisson distributed. We label the various realizations by means of the measures μ (a sum of Dirac δ-measures with unit strength), which are now the elements of a probability space $(M, P(\mathrm{d}\mu))$ with $P=P(\rho)$ the Poisson random measure (Kirsch 1989, Carmona and Lacroix 1990), which depends on ρ, the average scatterer density (its intensity in probabilistic terminology).

As discussed earlier (Tip 1994) averaged quantities can be formulated in terms of functions of the operator H in $\mathscr{K}=L^2(M, P(\mathrm{d}\mu);\mathscr{H})=\mathscr{H}\otimes L^2(M, P(\mathrm{d}\mu))=\mathscr{H}\otimes\mathscr{P}$, where H is the operator which equals $H(\mu)$ on the fibre μ. In a second paper with Maassen (Maassen and Tip 1993) it was shown that a unitary mapping can be made from $\mathscr{P}$ onto the symmetric Fock space $\mathscr{F}_{\mathrm{sym}}(\mathscr{H})$ over $\mathscr{H}$ such that averages now become vacuum expectation values. Since $N(E)$ is self-averaging (i.e. it has the same value for almost every $\mu \in M$) we can study it in this Fock space setting. Now the underlying Hilbert space is

$$\mathscr{K}_{\mathrm{F}} := \mathscr{H} \otimes \mathscr{F}_{\mathrm{sym}}(\mathscr{H}) \qquad \mathscr{F}_{\mathrm{sym}}(\mathscr{H}) := \mathbb{C} \oplus \mathscr{H} \oplus (\mathscr{H} \otimes \mathscr{H})_{\mathrm{sym}} \oplus \ldots$$
$$\mathscr{H} = \mathrm{L}^2(\mathbb{R}^d, \mathrm{d}x).$$

Written differently,

$$\mathscr{K}_{\mathrm{F}} = \bigoplus_{n=0}^{\infty} \mathscr{K}^{(n)} \qquad \mathscr{K}^{(0)} = \mathscr{H} \qquad \mathscr{K}^{(1)} = \mathscr{H} \otimes \mathscr{H}$$
$$\mathscr{K}^{(2)} = \mathscr{H} \oplus (\mathscr{H} \otimes \mathscr{H})_{\mathrm{sym}} \qquad \text{etc.}$$

Here $f = (f_0, f_1, \ldots, f_n, \ldots) \in \mathscr{K}_{\mathrm{F}}$ has norm squared

$$\|f\|^2 = \sum_{n=0}^{\infty} (n!)^{-1} (\|f_n\|_{L^2(\mathbb{R}^{d(n+1)})})^2.$$

The equivalent of H becomes

$$H_{\mathrm{F}} := \boldsymbol{P}^2 + \Phi(\varphi, \sqrt{\rho}) \tag{1.2}$$

where

$$\boldsymbol{P} = \bigoplus_{n=0}^{\infty} \boldsymbol{P}^{(n)} \qquad \text{with} \qquad \boldsymbol{P}^{(n)} = \sum_{j=0}^{n} \boldsymbol{p}_j$$

the total momentum operator acting in $\mathscr{K}^{(n)}$. Furthermore

$$\begin{aligned}\Phi(\varphi, \sqrt{\rho}) &= V(\varphi) + \sqrt{\rho} W(\varphi) + \rho\langle\varphi\rangle = \bigoplus_{n=1}^{\infty} V^{(n)}(\varphi) + \sqrt{\rho}\{b(\varphi) + b(\varphi)^*\} + \rho\langle\varphi\rangle \\ V^{(n)}(\varphi)(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) &= \sum_{j=1}^{n} \varphi(\boldsymbol{x}_j) \qquad \langle\varphi\rangle = \int \mathrm{d}\boldsymbol{x}\, \varphi(\boldsymbol{x})\end{aligned} \tag{1.3}$$

where $b(\varphi)$ and $b(\varphi)^*$ are annihilation and creation operators for $\varphi \in \mathscr{H}$. As shown in Maassen and Tip (1993) H_{F} is essentially self-adjoint for a large class of φs on a core $\mathscr{C}$ of smooth, compactly supported, finite particle vectors, i.e. elements of $\mathscr{K}_{\mathrm{F}}$ of the form $f = (f_0, f_1, \ldots, f_n, 0, 0, \ldots)$ with f_j in C_c^{∞}. We shall use the phrase nth layer for the space $\mathscr{K}^{(n)}$ as is usually done with Fock layers and drop the subscript F if no confusion can arise.

We shall show that for a class of repulsive potentials φ, the spectrum of H is absolutely continuous above a value $\zeta\rho\langle\varphi\rangle$, where ζ is a positive number, depending on d and φ. This entails the absolute continuity of $N(E)$ in the same range.

A well known method to establish absolute continuity is the complex dilatation one, originated by Combes *et al* (for a textbook version, see Reed and Simon 1978). It relies on relative compactness properties of the perturbations involved, giving control over the dilated essential spectrum. In the present case, although complex dilated closed operators $H_{\mathrm{F}}(\zeta)$ can be shown to exist, the term $\sqrt{\rho} W(\varphi, \zeta)$ is lacking relative compactness properties and we have no control over the spectrum. However, the corresponding first-order theory, developed by Mourre (for a textbook version of the Mourre method, see Cycon *et al* 1987), does work for a class of repulsive potentials, for instance of the form $\varphi(\boldsymbol{x}) = |\boldsymbol{x}|^{-\kappa}[1 + \boldsymbol{x}^2]^{-\lambda}$, with κ and λ positive in intervals depending on the dimension d. Our method of proof consists of a straightforward application of Mourre's commutator theorem. In fact we show that $\alpha > 0$ and $\zeta \in \mathbb{R}$ exist such that, as a form on $\mathscr{C}$,

$$[H, \mathrm{i}A] \geqslant \alpha\{H - \zeta\rho\langle\varphi\rangle\} \qquad A = \bigoplus_{n=0}^{\infty} A^{(n)}$$
$$A^{(n)} = \sum_{j=0}^{n} \tfrac{1}{2}(x_j \cdot p_j + p_j \cdot x_j) = \sum_{j=0}^{n} A_j. \tag{1.5}$$

We note that A is a direct generalization of the ordinary generator of dilatations in $L^2(\mathbb{R}^d)$. Denoting the spectral measure associated with H by $E(\Delta)$, Mourre's theorem states that H has purely absolutely continuous spectrum in any open interval Δ, for which $E(\Delta)[H, \mathrm{i}A]E(\Delta) \geqslant \gamma E(\Delta)$ with positive γ. In our case $\Delta \subset (\zeta\rho\langle\varphi\rangle, \infty)$ leading to absolute continuity of the spectrum of H in the interval $(\zeta\rho\langle\varphi\rangle, \infty)$.

Our results are rather restricted. In other cases treated by means of commutator methods (Reed and Simon 1978), A can be modified with the result that a larger class of potentials can be allowed (in particular the singular behaviour in $x=0$ is no longer required). Corresponding modifications of A in the present situation give unwieldy expressions involving additional terms on the right in the commutator in (1.4). In N-body systems they have compactness properties and can be handled (Cycon *et al* 1987) but here the situation is more complicated.

A second deficiency of our method is that no results are obtained in the interval $[0, \zeta\rho\langle\varphi\rangle]$ at the bottom of the spectrum. Here a Lifshitz tail develops (Kirsch 1989) and localization, if it occurs in the present type of random system, is expected to take place in an interval $[0, a]$ for $d \geqslant 3$. Also absolute continuity of $N(E)$ seems to play a role in a proof of localization (J M Combes and P Hislop, private communication).

2. Commutator estimates

In this section we derive the necessary commutator estimates needed for the application of the Mourre theory.

2.1. *The dilatation group on $L^2(\mathbb{R}^d)$*

Let $A_0^{(p)}$ be the generator of the dilatation group of isometries on $L^p(\mathbb{R}^d)$, $1 \leqslant p < \infty$, defined by

$$(U^{(p)}(\vartheta)f)(x) := (\exp[\mathrm{i}A_0^{(p)}\vartheta]f)(x) = \exp\left[\frac{d\vartheta}{p}\right]f(e^{\vartheta}x) \qquad f \in L^p(\mathbb{R}^d), \vartheta \in \mathbb{R}. \tag{2.1}$$

Denoting $A_0 = A_0^{(2)}$, we have, on a common dense set,

$$\mathrm{i}A_0 - \frac{d}{2} = \mathrm{i}A_0^{(p)} - \frac{d}{p} = x \cdot \partial_x \tag{2.2}$$

For $f \in L^2(\mathbb{R}^d)$ we have

$$g := [\mu - iA_0]^{-1} f = \begin{cases} \int_0^\infty du \exp[-(\mu - iA_0)u] f & \mu > 0 \\ -\int_0^\infty du \exp[(\mu - iA_0)u] f & \mu < 0 \end{cases} \tag{2.3}$$

so that, using (2.1) and making a change of variables ($x = |x|$, $e_x = x/x$),

$$g(x) = \begin{cases} x^{(\mu - d/2)} \int_x^\infty dy\, y^{(-1-\mu+d/2)} f(ye_x) & \mu > 0 \\ -x^{(\mu - d/2)} \int_0^x dy\, y^{(-1-\mu+d/2)} f(ye_x) & \mu < 0. \end{cases} \tag{2.4}$$

Thus $f(x) \geqslant 0$ a.e. $\Rightarrow g(x) \geqslant 0$ for $\mu > 0$ and $g(x) \leqslant 0$ for $\mu < 0$. Second, $f \in L^1 \cap L^p \Rightarrow g \in L^1 \cap L^p$, provided $\mu \neq -d/2$ and $\mu \neq (d/2) - (d/p)$. Even if $\mu = (d/2) - (d/p)$, $g \in L^1 \cap L^p$ provided $f \in L^1 \cap L^{p'}$, with $p' > p$, since now $g \in L^1 \cap L^{p'}$.

2.2. *Dilatation of H and commutators*

We now introduce the dilatation group $\{U(\vartheta) = \exp[i\vartheta A] | \vartheta \in \mathbb{R}\}$ on $\mathscr{H}_F$ by means of its action on the nth layer,

$$(U^{(n)}(\vartheta) f)(x_0, \ldots, x_n) = \exp[(n+1)\, d\vartheta/2] f(e^\vartheta x_0, \ldots, e^\vartheta x_n) \tag{2.5}$$

its generator being given in (1.4) when acting on elements of $\mathscr{C}$. We write H on $\mathscr{C}$ as

$$H = H_0 + \sqrt{\rho} W(\varphi) + \rho\langle\varphi\rangle \qquad H_0 = \bigoplus_{n=0}^{\infty} H_0^{(n)} \qquad H_0^{(n)} = (P^{(n)})^2 + V^{(n)}(\varphi). \tag{2.6}$$

Then, on the nth layer,

$$U^{(n)}(\vartheta) H_0^{(n)} U^{(n)}(\vartheta)^{-1} = \exp[-2\vartheta](P^{(n)})^2 + V^{(n)}(\varphi_\vartheta),\ \varphi_\vartheta(x) = \varphi(e^\vartheta x).$$

Secondly

$$(U(\vartheta) b(\varphi) U(\vartheta)^{-1} f)_{n-1}(x_0, \ldots, x_{n-1}) = \exp[d\vartheta/2](b(\varphi_\vartheta) f)_{n-1}(x_0, \ldots, x_{n-1})$$

and similar for $b(\varphi)^*$, so that

$$H(\vartheta) := U(\vartheta) H U(\vartheta)^{-1} = \exp[-2\vartheta] P^2 + V(\varphi_\vartheta) + \sqrt{\rho} \exp[d\vartheta/2] W(\varphi_\vartheta) + \rho\langle\varphi\rangle. \tag{2.7}$$

Proceeding formally, taking derivatives in $\vartheta = 0$, we obtain

$$\begin{aligned} C_1 &:= [H, iA] = 2P^2 - V(\eta) - \sqrt{\rho} W\left(\frac{d}{2}\varphi + \eta\right) \\ C_2 &:= [[H, iA], iA] = 4P^2 + V(\xi) + \sqrt{\rho} W\left(\left(\frac{d}{2}\right)^2 \varphi + d\eta + \xi\right) \end{aligned} \tag{2.8}$$

where $\eta = x \cdot \partial_x \varphi$ and $\xi = (x \cdot \partial_x)^2 \varphi$. (Note that $\langle\eta\rangle = -d\langle\varphi\rangle$, $\langle\xi\rangle = d^2\langle\varphi\rangle$). These formal results become exact as quadratic forms on $\mathscr{C}$ provided η and ξ are square integrable. In fact we can say more:

Proposition 2.1. Suppose that φ, η and ξ are contained in $L^1 \cap L^p(\mathbb{R}^d)$, $p=2$ for $d \leqslant 3$, $p>2$ for $d=4$ and $p=d/2$ for $d \geqslant 5$. Then C_1 and C_2 are essentially self-adjoint on $\mathscr{C}$.

Proof. The proof, which makes use of Nelson's commutator theorem, is the same as that of the essential self-adjointness of H on $\mathscr{C}$, as given in Maassen and Tip (1993). □

In the following sections 2.3–2.6 we shall assume that the conditions of the proposition are satisfied. Then the various f_js occurring there, being linear combinations of φ, η and ξ, are also in $L^1 \cap L^p$. In fact we shall reverse the procedure, demanding $f_j \in L^1 \cap L^p$.

2.3. *A positivity result*

In order to obtain the basic Mourre estimate we shall need the following lemma:

Lemma 2.2. Let $\psi \in L^1 \cap L^2(\mathbb{R}^d)$ and suppose $\psi(x) \geqslant 0$ for almost every x. Then, for $\lambda \in \mathbb{R}$, as a quadratic form on $\mathscr{C}$,

$$\Phi(\psi, \lambda) := V(\psi) + \lambda W(\psi) + \lambda^2 \langle \psi \rangle \geqslant 0. \tag{2.9}$$

Proof. For $f \in \mathscr{C}$ $(\Phi(\psi, \lambda)f, f) = (V(\psi)f, f) + 2\lambda \operatorname{Re}(b(\psi)f, f) + \lambda^2 \langle \psi \rangle \| f \|^2$. Now (note that the sums below are actually finite, f being contained in $\mathscr{C}$)

$$\begin{aligned}
&2\lambda \operatorname{Re}(b(\psi)f, f) + \lambda^2 \langle \psi \rangle \|f\|^2 \\
&= 2\lambda \operatorname{Re} \sum_{n=0}^{\infty} (n!)^{-1} \int \mathrm{d}x_0 \ldots \mathrm{d}x_{n+1} \psi(x_{n+1}) f_{n+1}(x_0, \ldots, x_{n+1}) \\
&\quad \times \overline{f_n(x_0, \ldots, x_n)} \\
&\quad + \lambda^2 \sum_{n=0}^{\infty} (n!)^{-1} \int \mathrm{d}x_0 \ldots \mathrm{d}x_{n+1} \psi(x_{n+1}) | f_n(x_0, \ldots, x_n)|^2 \\
&= \sum_{n=0}^{\infty} (n!)^{-1} \int \mathrm{d}x_0 \ldots \mathrm{d}x_{n+1}\, \psi(x_{n+1}) | f_{n+1}(x_0, \ldots, x_{n+1}) \\
&\quad + \lambda f_n(x_0, \ldots, x_n)|^2 \\
&\quad - \sum_{n=0}^{\infty} (n!)^{-1} \int \mathrm{d}x_0 \ldots \mathrm{d}x_{n+1}\, \psi(x_{n+1}) | f_{n+1}(x_0, \ldots, x_{n+1})|^2 \\
&= \text{first term} - \sum_{n=1}^{\infty} [(n-1)!]^{-1} \int \mathrm{d}x_0 \ldots \mathrm{d}x_n\, \psi(x_n) | f_n(x_0, \ldots, x_n)|^2 \\
&= \text{first term} - \sum_{n=1}^{\infty} (n!)^{-1} \int \mathrm{d}x_0 \ldots \mathrm{d}x_n \left\{ \sum_{j=1}^{n} \psi(x_j) \right\} | f_n(x_0, \ldots, x_n)|^2 \\
&= \text{first term} - (V(\psi)f, f)
\end{aligned}$$

and the first term is non-negative. □

Remark. For $\lambda > 0$ the statement also follows from the positivity of $\langle \varphi, \mu \rangle$ in $\mathscr{K}$.

2.4. The Mourre estimate for C_1

Let $\varphi \geqslant 0$. We write, as a form on $\mathscr{C}$,

$$C_1 = 2P^2 + \alpha\Phi(\varphi, \sqrt{\rho}) + R \qquad \alpha > 0. \tag{2.10}$$

Then

$$R = V(-\alpha\varphi - \eta) + \sqrt{\rho}W\left(-\left(\alpha + \frac{d}{2}\right)\varphi - \eta\right) - \alpha\rho\langle\varphi\rangle. \tag{2.11}$$

Since

$$2P^2 + \alpha\Phi(\varphi, \sqrt{\rho}) \geqslant \alpha_0 H \qquad \alpha_0 = \min\{2, \alpha\} \tag{2.12}$$

we have a Mourre estimate provided $R \geqslant \beta$ for some $\beta \in \mathbb{R}$. The idea is now to see whether R can be written as a sum of positive Φs and a real constant. Thus we put

$$R = \Phi(-a\eta - b\varphi, \lambda_1\sqrt{\rho}) + \Phi(-f\eta - g\varphi, \lambda_2\sqrt{\rho}) + h\rho\langle\varphi\rangle. \tag{2.13}$$

Then $C_1 \geqslant \alpha_0 H + h\rho\langle\varphi\rangle\}$. Comparing (2.11) and (2.13) we obtain

$$(\lambda_1 - \lambda_2)a = 1 - \lambda_2 \qquad (\lambda_1 - \lambda_2)b = (1 - \lambda_2)\alpha + \frac{d}{2}$$

$$h = -\frac{d}{2}[(\lambda_1 + \lambda_2 - 2\lambda_1\lambda_2)] - \alpha(1 - \lambda_1)(1 - \lambda_2)(d - \alpha).$$

Note that $\lambda_1 = \lambda_2$ gives $d = 0$, a contradiction. A tedious calculation reveals that no generality is lost by setting $f = 0$. Then

$$a = 1 \qquad \lambda_1 = 1 \qquad b = \alpha + \frac{d}{2}[1 - \lambda_2]^{-1} \qquad g = -\frac{d}{2}[1 - \lambda_2]^{-1} \qquad h = -\frac{d}{2}[1 - \lambda_2].$$

Thus, with $\varepsilon = 1 - \lambda_2$,

$$R = \Phi\left(-\eta - \alpha\varphi - \frac{d}{2\varepsilon}\varphi, \sqrt{\rho}\right) + \Phi\left(\frac{d}{2\varepsilon}\varphi, (1 - \varepsilon)\sqrt{\rho}\right) - \frac{d\varepsilon}{2}\rho\langle\varphi\rangle.$$

Both Φs are non-negative provided $\varepsilon > 0$ and

$$f_1 := -\eta - \alpha\varphi - \frac{d}{2\varepsilon}\varphi \geqslant 0$$

or

$$\left[\frac{d}{p} - \alpha - \frac{d}{2\varepsilon} - -\mathrm{i}A_0^{(p)}\right]\varphi = f_1 \geqslant 0.$$

Hence, formally,

$$\varphi = \left[\frac{d}{\rho} - \alpha - \frac{d}{2\varepsilon} - \mathrm{i}A_0^{(\rho)}\right]^{-1} f_1 = [c_1 - \mathrm{i}A_0]^{-1} f_1. \tag{2.14}$$

By our results in subsection 2.1. this becomes true by taking a non-negative function from $L^1 \cap L^p$ for f_1 and requiring that

$$c_1 = \frac{d}{2} - \alpha - \frac{d}{2\varepsilon} > 0$$

(this implies that $\varphi \in L^1$) and

$$\frac{d}{p} - \alpha - \frac{d}{2\varepsilon} \neq 0.$$

(Recall that we may have equality in the last expression provided $f_1 \in L^1 \cap L^{p'}$ with $p' > p$.) Note also that $c_1 > 0$ implies $\alpha < d/2$. Thus we have $\alpha_0 = \alpha$ for $d \leqslant 3$. Now, as a form on $\mathscr{C}$,

$$C_1 \geqslant \alpha_0 H - \frac{d\varepsilon}{2} \rho\langle\varphi\rangle = \alpha_0\{H - \zeta\rho\langle\varphi\rangle\} \qquad \zeta = \frac{d\varepsilon}{2\alpha_0}. \tag{2.15}$$

For $d \leqslant 3$ and given c_1 ζ attains the minimal value

$$\zeta_{\min} = d^2\left(\frac{d}{2} - c_1\right)^{-2} \geqslant 4 \quad \text{for} \quad \alpha = \frac{1}{2}\left(\frac{d}{2} - c_1\right).$$

In order to apply the Mourre theory we need some further properties of C_1 and C_2 so that they have certain boundedness properties as transformations in the scale of spaces generated by H (Cycon *et al* 1987). We shall show that under some additional restrictions on φ this is actually the case. We start with C_1.

2.5. *Bounding C_1 from above*

We try to bound C_1 from the other side: $\gamma_0 H + \beta \geqslant C_1$. Now, with $\mu \neq 0$,

$$\begin{aligned} C_1 &= 2P^2 + \gamma\Phi(\varphi, \sqrt{\rho}) - V(\gamma\varphi + \eta) - \sqrt{\rho} W\left(\left(\gamma + \frac{d}{2}\right)\varphi + \eta\right) - \gamma\rho\langle\varphi\rangle \\ &= 2P^2 + \gamma\Phi(\varphi, \sqrt{\rho}) + \Phi(-(\gamma-\mu)\varphi - \eta, \sqrt{\rho}) \\ &\quad + \Phi\left(-\mu\varphi, \left(1 + \frac{d}{2\mu}\right)\sqrt{\rho}\right) - \frac{d^2}{4\mu}\rho\langle\varphi\rangle. \end{aligned}$$

Since $2P^2 + \gamma\Phi(\varphi, \sqrt{\rho}) \leqslant \gamma_0 H$, $\gamma_0 = \max\{2, \gamma\}$, $\gamma > 0$, the desired result follows if the last two Φs are non-positive, i.e. if $(\mu - \gamma)\varphi + \eta = f_2 \geqslant 0$ and $\mu > 0$. Thus

$$\varphi = \left[\gamma - \mu - \frac{d}{2} + \mathrm{i}A_0\right]^{-1} f_2 = [c_2 + \mathrm{i}A_0]^{-1} f_2 \tag{2.16}$$

where $\varphi \geqslant 0$ and $\varphi \in L^1 \cap L^p$ hold, provided $f_2 \in L^1 \cap L^p$ and

$$\frac{d}{2} < \gamma - \mu \neq d \qquad \frac{d}{p}. \tag{2.17}$$

Together with (2.15) we now have, as forms on $\mathscr{C}$,

$$\gamma_0 H - \frac{d^2}{4\mu}\rho\langle\varphi\rangle \geqslant C_1 \geqslant \alpha_0 H - \frac{\mathrm{d}\varepsilon}{2}\rho\langle\varphi\rangle \tag{2.18}$$

and it follows that $[1+H]^{-1/2}C_1[1+H]^{-1/2}$ defines a bounded operator on $\mathscr{H}_{\mathrm{F}}$. In particular $E(\Delta)C_1E(\Delta)\in\mathscr{B}(\mathscr{H}_{\mathrm{F}})$ and we have

$$E(\Delta)C_1E(\Delta) > \gamma E(\Delta) \qquad \gamma>0 \qquad \Delta\subset(\zeta\rho\langle\varphi\rangle, \infty). \tag{2.19}$$

2.6. Bounding C_2 from above

We write, with $\delta>0$, $\nu\neq 0$,

$$\begin{aligned} C_2 &= 4P^2 + V(\xi) + \sqrt{\rho}W\left(\frac{d^2}{4}\varphi + \mathrm{d}\eta + \xi\right) \\ &= 4P^2 + \delta\Phi(\varphi, \sqrt{\rho}) + \Phi\left(-\left(\delta + \frac{\nu d}{4}\right)\varphi - \nu\eta + \xi, \sqrt{\rho}\right) \\ &\quad + \Phi\left(\nu\left(\frac{d}{4}\varphi + \eta\right), \left(1 + \frac{d}{\nu}\right)\sqrt{\rho}\right) + \kappa\rho\langle\varphi\rangle \end{aligned} \tag{2.20}$$

where

$$\kappa = \left(\frac{d}{2}\right)^2\left(\frac{3d}{\nu} + 2\right).$$

The two last Φs are non-positive if

$$-\left(\delta + \frac{d}{4}\nu\right)\varphi - \nu\eta + \xi = -f_3 \leqslant 0 \qquad \nu\left(\frac{d}{4}\varphi + \eta\right) = -f_4 \leqslant 0. \tag{2.21}$$

Thus

$$\varphi = \nu^{-1}\left[\frac{d}{4} - \mathrm{i}A_0\right]^{-1} f_4 = \nu^{-1}[c_4 - \mathrm{i}A_0]^{-1} f_4$$

giving $\nu>0$ and $\varphi\in L^1$. Also $\varphi\in L^p$ for $p\neq 4$ but for $p=4$ we have to require that $f_4\in L^{p'}, p'>p$. (Alternatively we can make a slightly different decomposition in a sum of Φs.) Second,

$$f_3 = [c_+ - \mathrm{i}A_0][c_- + \mathrm{i}A_0]\varphi \qquad c_\pm = [\delta + \tfrac{1}{4}(\nu^2 + \nu d)]^{1/2} \pm \frac{\nu + d}{2}$$

or

$$\varphi = [c_+ - \mathrm{i}A_0]^{-1}[c_- + \mathrm{i}A_0]^{-1} f_3$$

and $\varphi\geqslant 0$ requires $c_\pm>0$, i.e. $\delta>(d/4)\,(d+\nu)$. Then

$$C_2 \leqslant \delta_0 H + \kappa\rho\langle\varphi\rangle \qquad \delta_0 = \max\{4, \delta\}. \tag{2.22}$$

We meet the $L^1 \cap L^p$-requirement by demanding

$$\left(\frac{d}{q}\right)^2 + \left(\frac{d}{q} - \frac{d}{4}\right)\nu \neq \delta \qquad q = 1, p.$$

2.7. Bounding C_2 from below

Now the last two Φs in (2.21) must be non-negative, i.e. the signs change in (2.21). Replacing δ, ν and κ by δ', $-\nu'$ and κ', we have

$$-\left(\delta' - \frac{d}{4}\nu'\right)\varphi + \nu'\eta + \xi = f_5 \geqslant 0 \qquad -\nu'\left(\frac{d}{4}\varphi + \eta\right) = f_6 \geqslant 0. \tag{2.23}$$

The second results in $\nu' > 0$ and the first can be written as

$$\left\{\left[\frac{d-\nu'}{2} - \mathrm{i}A_0\right]^2 + \frac{\nu'}{4}(d-\nu') - \delta'\right\}\varphi = f_5 \geqslant 0$$

leading to

$$\varphi = [c'_+ - \mathrm{i}A_0]^{-1}[c'_- - \mathrm{i}A_0]^{-1} f_5 \qquad c'_\pm = \frac{d-\nu'}{2} \pm [\delta' + \tfrac{1}{4}(\nu')^2 - \tfrac{1}{4}\nu' d]^{1/2}$$

for

$$\delta' + \tfrac{1}{4}(\nu')^2 - \tfrac{1}{4}\nu' d > 0.$$

This, together with the requirement $c'_\pm > 0$, gives

$$\frac{d}{4}(d-\nu') > \delta' > \frac{d}{4}(d-\nu').$$

We shall disregard the other case. In this situation $C_2 \geqslant \delta'_0 H + \kappa' \rho \langle\varphi\rangle$, $\delta'_0 = \min\{4, \delta'\}$. Combining this with (2.22):

$$\delta_0 H + \kappa\rho\langle\varphi\rangle \geqslant C_2 \geqslant \delta'_0 H + \kappa'\rho < \varphi. \tag{2.24}$$

Hence $[1+H]^{-1/2} C_2 [1+H]^{-1/2} \in \mathscr{B}(\mathscr{H}_{\mathrm{F}})$. Note that this would already follow if $\delta'_0 = 0$, i.e. C_2 merely bounded from below. Instead we see that we have actually obtained a 'multiple commutator' inequality (for the latter, see Jensen *et al* 1984).

2.8. The allowed class of φ's

One way to satisfy the various requirements in subsections 2.4–2.7 is by taking φ of the form

$$\varphi = \left(\prod_\gamma [c_\gamma - \mathrm{i}A_0]^{-1}\right)\left(\prod_\delta [c_\delta + \mathrm{i}A_0]^{-1}\right) f \qquad c_\alpha > 0 \qquad f \geqslant 0 \qquad f \in L^1 \cap L^p.$$

We can, for example, satisfy subsections 2.4–2.7 by taking

$$\begin{aligned} &\varphi = [\nu_1 - \mathrm{i}A_0]^{-1}[\nu_2 + \mathrm{i}A_0]^{-1} f \\ &\nu_1 \leqslant \min\left\{c_1, c_4 = \frac{d}{4}, c_+\right\} = \min\left\{c_1, \frac{d}{4}\right\} \qquad \nu_2 \leqslant \min\{c_2, c_-\}. \end{aligned} \tag{2.25}$$

Then the corresponding fs are non-negative. For instance

$$f_3=[c_+-\mathrm{i}A_0][c_-+\mathrm{i}A_0]\varphi=\{[c_+-\nu_1][\nu_1-\mathrm{i}A_0]^{-1}+1\}\{[c_--\nu_2][\nu_2+\mathrm{i}A_0]^{-1}+1\}f\geqslant 0.$$

Subsection 2.7 can now be satisfied by requiring $f=[\nu_3-\mathrm{i}A_0]^{-1}h$, with suitable ν_3 and further adjustment of ν_1 or, alternatively, by a suitable choice of δ' and ν'.

A different, simpler, approach is to observe that the various positivity requirements can be met if $\eta\leqslant 0$, $k_1\varphi(x)\leqslant|\eta(x)|\leqslant k_2\varphi(x)$ and $|\xi(x)|\leqslant k_3\varphi(x)$ for a.e. x and positive k_js. For example

$$\varphi(x)=x^{-\kappa}[1+x^2]^{-\lambda} \qquad \max\left\{\frac{d}{4}, \alpha+\frac{d}{2\varepsilon}\right\}\leqslant\kappa<\frac{d}{p} \qquad \kappa+2\lambda>d \tag{2.26}$$

satisfies all our requirements for suitably chosen ν, δ, ν' and δ'. Here $\zeta_{\min}=(d/\kappa)^2$.

3. Absolute continuity results

In section 2 we obtained the necessary information to state the following theorem:

Theorem 3.1. There exist (non-negative) $\varphi\in L^1\cap L^p(\mathbb{R}^d)$, $p=2$ for $d\leqslant 3$, $p>2$ for $d=4$ and $p=d/2$ for $d=5$, such that the spectrum of H is absolutely continuous in an interval $(\zeta\rho\langle\varphi\rangle, \infty)$, $\zeta>0$. An example is given by (2.26).

Proof. According to our discussion in section 2 there exist $0\leqslant\varphi\in L^1\cap L^p(\mathbb{R}^d)$ such that $[1+H]^{-1/2}C_1[1+H]^{-1/2}$ and $[1+H]^{-1/2}C_2[1+H]^{-1/2}$ define bounded operators on $\mathscr{H}_{\mathrm{F}}$ with C_1 being the estimate $E(\Delta)C_1E(\Delta)>\gamma E(\Delta)$, $\gamma>0$, $\Delta\subset(\zeta\rho\langle\varphi\rangle, \infty)$, $\zeta>0$. Thus the hypotheses 1, 2, 2′ and 4 on p. 62 of Cycon *et al* (1987) referred to as CFKS are satisfied. Thus corollary 4.10 of CFKS applies provided their lemma 4.12 holds. In CFKS this lemma is proved using their hypothesis 3 which is irrelevant in the present case, there being no natural decomposition $H=H_0+V$ with the required properties. However, the statement of lemma 4.12 also follows under the assumptions $C_1\in\mathscr{B}(\mathscr{H}_k)$ and $U(\vartheta)$ defines a strongly continuous, exponentially bounded semi-group on $\mathscr{H}_k$, $k=\pm 1$, where $\{\mathscr{H}_k|k\in\mathbb{N}\}$ is the scale of spaces associated with H. The first of these was already verified whereas the second follows from the following considerations:

For $\psi\geqslant 0$ and $\lambda\in\mathbb{R}$ we have

$$\Phi(\psi,\lambda)\geqslant\mu\Phi(\psi,\lambda/\mu)-\mu^{-1}(1-\mu)\lambda^2\langle\psi\rangle \qquad \mu\in(0,1]$$

$$\Phi(\psi,\gamma\lambda)\geqslant\mu\Phi(\psi,\lambda)-\mu(1-\mu)^{-1}\lambda^2\langle\psi\rangle \qquad \mu\in[0,1) \qquad \gamma\in\mathbb{R}.$$

Now let $\psi\geqslant 0$ be repulsive and $\vartheta\geqslant 0$. Then $\psi_\vartheta(x)\leqslant\psi(x)$ and the first of the above relations with $\mu=\exp[-\mathrm{d}\vartheta/2]$ together with lemma 2.2 results in

$$\Phi_\vartheta(\psi,\lambda)\leqslant\exp[\mathrm{d}\vartheta/2]\{\Phi(\psi,\lambda)+(\exp[\mathrm{d}\vartheta/2]-1)\lambda^2\langle\psi\rangle\}$$

whereas, if in addition $\psi_\vartheta(x)\geqslant\exp[-\alpha\vartheta]\psi(x)$ for $\vartheta\geqslant 0$ and some $\alpha>0$, the second relation with $\mu=\gamma^1=\exp[-\mathrm{d}\vartheta/2]$ and lemma 2.2 lead to

$$\Phi_\vartheta(\psi,\lambda)\geqslant\exp[-(\alpha+d/2)\vartheta]\Phi(\psi,\lambda)-(\exp[\mathrm{d}\vartheta/2]-1)\exp\{-\mathrm{d}\vartheta]\lambda^2\langle\psi\rangle.$$

Combining results we have, for $\vartheta \geq 0$ and $\beta = \max\{2, \alpha + d/2\}$,

$$\exp[-\beta\vartheta]H - (\exp[\mathrm{d}\vartheta/2]-1)\exp[-\mathrm{d}\vartheta]\rho\langle\varphi\rangle$$
$$\leq H(\vartheta) \leq \exp[\mathrm{d}\vartheta/2]\{H + (\exp[\mathrm{d}\vartheta/2]-1)\rho\langle\varphi\rangle\}$$

with a similar result for $\vartheta < 0$ obtained by sandwiching this expression between $U(\vartheta)^{-1}$ and $U(\vartheta)$. Now we can conclude that $\mathscr{D}(H(\vartheta)^{1/2}) = \mathscr{D}(H^{1/2})$ and that $U(\vartheta)$ is exponentially bounded on $\mathscr{H}_{\pm 1}$. □

We now turn to the integrated density of states measure $\nu(\Delta)$ (the integrated density of states $N(E)$ is given by $N(E) = \nu((-\infty, E])$ $(=\nu([0, E])$ in our case). In the present situation with non-negative potentials, proposition VI.1.3 of Carmona and Lacroix (1990) applies so that we have, for square integrable, strictly positive, continuous f with unit L^2-norm and Δ a bounded Borel set in $\mathbb{R}$

$$\nu(\Delta) = \int P(\mathrm{d}\mu)\, \mathrm{tr}_{\mathscr{H}} f(x) E_\mu(\Delta) f(x) = \mathrm{tr}_{\mathscr{K}} f(x) E(\Delta) f(x) \{\mathrm{I}_{\mathscr{H}} \otimes (\cdot\,, \omega_0)_{\mathscr{F}} \omega_0\} \tag{3.1}$$

where ω_0 is the vacuum state in $\mathscr{F}$ and $E_\mu(\cdot)$ and $E(\cdot)$ are the spectral decompositions of $H(\mu)$ and H, respectively. (The ergodicity of the Poisson process guarantees that the density of states measure has the same value for a.e. μ. Thus it equals its integral over $P(\mathrm{d}\mu)$.) Let $\{u_j\}$ be an orthonormal basis for $\mathscr{H}$. Then

$$\nu(\Delta) = \lim_{n\to\infty} \nu^{(n)}(\Delta) \qquad \nu^{(n)}(\Delta) = \sum_{j=1}^{n} \nu_j(\Delta)$$
$$\nu_j(\Delta) = (E(\Delta f u_j \otimes \omega_0, f u_j \otimes \omega_0)_{\mathscr{K}}. \tag{3.2}$$

Now let $\Delta \subset (\zeta\rho\langle\varphi\rangle, \infty)$ be an interval. Using the absolute continuity of $E(\Delta)$,

$$\nu_j(\Delta) = \int_\Delta \mathrm{d}\lambda\, g_j(\lambda)$$

with $g_j \in L^1(\Delta)$, non-negative, and

$$\nu^{(n)}(\Delta) = \int_\Delta \mathrm{d}\lambda\, g^{(n)}(\lambda)$$

where

$$g^{(n)}(\lambda) = \sum_{j=1}^{n} g_j(\lambda).$$

We note that $\{g^{(n)}\}$ is a Cauchy sequence with L^1-limit g and that

$$\nu(\Delta) = \int_\Delta \mathrm{d}\lambda\, g(\lambda).$$

Thus we have shown:

Theorem 3.2: Under the assumptions of theorem 3.1 the integrated density of states measure $\nu(\Delta)$ is absolutely continuous in $(\zeta\rho\langle\varphi\rangle, \infty)$.

4. Discussion

Observing that H does not depend on the variable x_0 we obtain a direct integral decomposition by means of the Fourier map (see Tip 1994, Maassen and Tip 1993 for details). Thus

$$H=\int_{\mathbb{R}^d}^{\oplus} \mathrm{d}\boldsymbol{k}\; H(\boldsymbol{k}) \qquad H(\boldsymbol{k})=(\boldsymbol{q}+\boldsymbol{k})^2+\Phi(\varphi,\sqrt{\rho}) \tag{4.1}$$

with $H(\boldsymbol{k})$ acting in $\mathscr{F}=\mathscr{F}_{\rm sym}(\mathscr{H})$ and

$$\boldsymbol{q}=\bigoplus_{n=1}^{\infty}\boldsymbol{q}^{(n)} \qquad \text{with} \qquad \boldsymbol{q}^{(n)}=\sum_{j=1}^{n}\boldsymbol{p}_j$$

the total momentum operator acting in $\mathscr{F}^{(n)}$, the nth Fock layer. With

$$A=\bigoplus_{n=1}^{\infty}A^{(n)} \qquad A^{(n)}=\sum_{j=1}^{n}\tfrac{1}{2}(\boldsymbol{x}_j\cdot\boldsymbol{p}_j+\boldsymbol{p}_j\cdot\boldsymbol{x}_j)=\sum_{j=1}^{n}A_j \tag{4.2}$$

we obtain, for $0<\alpha<2$, working things out as before

$$\begin{aligned}C_1(\boldsymbol{k}) &:= [H(\boldsymbol{k}),\mathrm{i}A]=2(\boldsymbol{q}+\boldsymbol{k})\cdot\boldsymbol{q}+[\Phi,\mathrm{i}A]\geqslant\alpha(\boldsymbol{q}+\boldsymbol{k})^2-[2-\alpha]^{-1}\boldsymbol{k}^2+[\Phi,\mathrm{i}A]\\ &\geqslant\alpha\left\{H(\boldsymbol{k})-\frac{\mathrm{d}\varepsilon}{2\alpha}\rho\langle\varphi\rangle-\alpha^{-1}[2-\alpha]^{-1}k^2\right\}\end{aligned} \tag{4.3}$$

leading to absolute continuity of the spectrum of $H(\boldsymbol{k})$ in an interval

$$\Delta=\left(\frac{\mathrm{d}\varepsilon}{2\alpha}\rho\langle\varphi\rangle+\alpha^{-1}[2-\alpha]^{-1}k^2,\infty\right). \tag{4.4}$$

Thus we have a weaker result on the fibres, not even uniform in $\boldsymbol{k}$ (on the other hand it is not *a priori* evident that $H(\boldsymbol{k})$ should have any a.c. properties at all). We can look upon the above result from a different angle by noting that $H(\boldsymbol{k})$ can be written as follows: Let $P=(\cdot,\omega_0)\omega_0$ be the projector upon layer 0, $Q=1-P$ and $\varphi^{(1)}=(0,\varphi,0,0,\ldots)$. Then

$$H(\boldsymbol{k})=k^2P+QH(\boldsymbol{k})Q+\sqrt{\rho}\{(\cdot,\omega_0)\varphi^{(1)}+(\cdot,\varphi^{(1)})\omega_0\}=H_0(\boldsymbol{k})+\sqrt{\rho}W^{(1)} \tag{4.5}$$

where $H_0(\boldsymbol{k})=k^2P+QH(\boldsymbol{k})Q$ has the eigenvalue k^2 and $\sqrt{\rho}W^{(1)}$ is finite-dimensional. If $QH(\boldsymbol{k})Q$ has purely a.c. spectrum in a neighbourhood of k^2, we expect that the perturbation $\sqrt{\rho}W^{(1)}$ will remove the eigenvalue. It is intriguing to note that an unsymmetrized extension $H_{\rm us}(\boldsymbol{k})$ of $H(\boldsymbol{k})$ exists for which lemma 2.2 applies with the result that $QH_{\rm us}Q$ has purely a.c. spectrum in

$$\left(\frac{\mathrm{d}\varepsilon}{2\alpha}\rho\langle\varphi\rangle,\infty\right).$$

In this extension Φ is replaced by

$$\Phi_{\rm us}(\varphi,\sqrt{\rho})=V_{\rm us}(\varphi)+\sqrt{\rho}\{a(\varphi)+a(\varphi)^*\}+\rho\langle\varphi\rangle$$

where

$$(V_{us}(\varphi)f)_n(x_1, \ldots, x_n) = n\varphi(x_n)f_n(x_1, \ldots, x_n)$$

$$(a(\varphi)^*f)_n(x_1, \ldots, x_n) = nf_{n-1}(x_1, \ldots, x_{n-1})\varphi(x_n).$$

Then $H(k) = \Pi H_{us}(k)\Pi$, where Π is the symmetrization projector:

$$(\Pi f)_n(x_1, \ldots, x_n) = (n!)^{-1} \sum_{\text{perm}} f_n(x_{j_1}, \ldots, x_{j_n}).$$

Now, using $A(k) = A + k \cdot X_1$, $(X_1 f)_n(x_1, \ldots, x_n) = x_1 f_n(x_1, \ldots, x_n)$, instead of A, we obtain the relevant Mourre estimate for $QH_{us}Q$. Note that $k \cdot X_1$ commutes with Φ_{us}. However, since Π does not reduce $H_{us}(k)$ we cannot draw conclusions about $QH(k)Q$ from this result. In a two-layer approximation (see Maassen and Tip 1993), on the other hand, the generic situation is such that k^2 changes into a resonance.

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